

# TRUDINGER-MOSER INEQUALITY WITH REMAINDER TERMS

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ABSTRACT. The paper gives the following improvement of the Trudinger-Moser inequality:

$$(0.1) \quad \sup_{\int_{\Omega} |\nabla u|^2 dx - \psi(u) \leq 1, u \in C_0^\infty(\Omega)} \int_{\Omega} e^{4\pi u^2} dx < \infty, \quad \Omega \in \mathbb{R}^2,$$

related to the Hardy-Sobolev-Mazya inequality in higher dimensions. We show (0.1) with  $\psi(u) = \int_{\Omega} V(x)u^2 dx$  for a class of  $V > 0$  that includes

$$V(r) = \frac{1}{4r^2(\log \frac{1}{r})^2 \max\{\sqrt{\log \frac{1}{r}}, 1\}},$$

which refines two previously known cases of (0.1) proved by Adimurthi and Druet [3] and by Wang and Ye [24]. In addition, we verify (0.1) for  $\psi(u) = \lambda \|u\|_p^2$ , as well as give an analogous improvement for the Onofri-Beckner inequality.

## 1. INTRODUCTION.

The Trudinger-Moser inequality ([25, 18, 20, 23, 15])

$$(1.1) \quad \sup_{\int_{\Omega} |\nabla u|^2 dx \leq 1, u \in C_0^\infty(\Omega)} \int_{\Omega} e^{4\pi u^2} dx < \infty,$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded domain, is an analog of the limiting Sobolev inequality in  $\mathbb{R}^N$  with  $N \geq 3$ :

$$(1.2) \quad \sup_{\int_{\mathbb{R}^N} |\nabla u|^2 dx \leq 1, u \in C_0^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u|^{2^*} dx < \infty, \quad 2^* = \frac{2N}{N-2}.$$

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We recall that restriction of inequalities involving the gradient norm to bounded domains is of essence when  $N = 2$ , since the completion of  $C_0^\infty(\mathbb{R}^2)$  in the gradient norm is not a function space, and, moreover, since  $\int_B |\nabla u|^2 dx$  on the unit disk  $B \subset \mathbb{R}^2$  coincides with the quadratic form of the Laplace-Beltrami operator on the hyperbolic plane (a *complete non-compact* Riemannian manifold) when expressed in the coordinates of the Poincaré disk.

Both limiting Trudinger-Moser and Sobolev inequalities are optimal in the sense that they are false for any nonlinearity that grows as  $s \rightarrow \infty$  faster than  $e^{4\pi s^2}$  resp  $s^{2^*}$ . Inequality (1.2) is also false if the nonlinearity  $|u|^{2^*}$  is multiplied by an unbounded radial monotone function, although (1.1) on the unit disk holds also when the integrand is replaced by  $\frac{e^{4\pi u^2}-1}{(1-r)^2}$  ([4, 10]).

This paper studies another refinement of (1.1), whose analogy in the case  $N \geq 3$  is the Mazya's refinement of (1.2), known as Hardy-Sobolev-Mazya inequality ([15]):

$$(1.3) \quad \sup_{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} V_m(x) u^2 dx \leq 1, u \in C_0^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u|^{2^*} dx < \infty,$$

where

$$V_m(x) = \left( \frac{m-2}{2} \right)^2 \frac{1}{|x_1 + \dots + x_m|^2}, \quad m = 1, \dots, N-1.$$

It is false when  $m = N$ , and similarly, inequality (0.1) does not hold with  $\psi(u) = \int_B V(|x|) u^2 dx$ , if  $V$  is the two-dimensional counterpart of the Hardy's radial potential, the Leray's potential

$$V_{\text{Leray}}(r) = \frac{1}{4r^2 (\log \frac{1}{r})^2}.$$

When  $\psi(u) = \int_\Omega V(x) u^2 dx$ , inequality (0.1) has been already established for two specific potentials  $V$ . In one case, proved by Adimurthi and Druet [3],  $V(x) = \lambda < \lambda_1$ , and  $\lambda_1$  is the first eigenvalue of the Dirichlet Laplacian in  $\Omega$ . Note only that the inequality stated as a main result in [3] is formally weaker, but it immediately implies (0.1) with  $V(x) = \lambda < \lambda_1$  via an elementary argument). It was conjectured by Adimurthi [2] that the inequality remains valid whenever one replaces  $\int_\Omega \lambda u^2 dx$  with a general weakly continuous functional  $\psi$ , as long as  $\|\nabla u\|_2^2 - \psi(u) > 0$  for  $u \neq 0$ . Another known case of the inequality

(0.1), with  $\psi(u) = \int_B \frac{u^2}{(1-r^2)^2} dx$ , is due to Wang and Ye [24]. Note that the result of Wang and Ye involves a non-compact remainder term, and that via conformal maps it extends to general domains.

In deciding about the natural counterpart of the Hardy-Sobolev-Mazya inequality in the two-dimensional case, we have to make a choice, which is insignificant in the case  $N \geq 3$ , between using the functional  $\int e^{4\pi u^2}$  and the Orlicz norm  $\|u\|_{\text{Orl}}$  associated with the integrand (in terms of the standard definition, with the function  $e^{4\pi s^2} - 1$ ). The difference between the case  $N \geq 3$  and  $N = 2$  is in the fact that (1.3) can be equivalently rewritten as

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} V_m(x) u^2 dx \geq C \|u\|_{2^*}^2,$$

while from

$$(1.4) \quad \int_{\Omega} |\nabla u|^2 dx - \psi(u) \geq C \|u\|_{\text{Orl}}^2$$

for  $N = 2$  inequality (0.1) does not follow, and instead one has its weaker version, with the bound on  $\int_{\Omega} e^{Cu^2} dx$  with *some*  $C$ . In particular, in the case of Adimurthi-Druet,  $V(x) = \lambda < \lambda_1$ , inequality (1.4) is completely trivial while their actual result is very sharp. This example explains why we, following Wang and Ye, treat (0.1), and not (1.4), as a natural counterpart of (1.3).

The objective of this paper is to prove the inequality (0.1) with the more general (and in particular, stronger) remainder term  $\psi(u)$  than in the two known cases. In Section 2 we study the case  $p = 2$  and the radial potential on a unit disk, in Section 3 we extend the result to general bounded domains and to the values  $p > 2$ . In Section 4 we give corollaries to the inequalities, prove a related refinement of Onofri-Beckner inequality, and list some open problems.

In what follows,  $B$  will denote an open unit disk,  $\|\cdot\|_p$  will mean the  $L^p(\Omega)$ -norm when the domain is specified, and the subspace of radial functions of, say, Sobolev space  $H_0^1(B)$  will be denoted  $H_{0,\text{rad}}^1(B)$ .

## 2. REMAINDER WITH A SINGULAR POTENTIAL.

**Ground state alternative.** We summarize first some relevant results on positive elliptic operators with singular potentials, drawing upon [19].

Let  $\Omega \subset \mathbb{R}^N$  be a domain, and let  $V$  be a continuous function in  $\Omega$ . We consider the functional

$$(2.1) \quad Q_V(u) = \int_{\Omega} |\nabla u|^2 dx - \psi(u), \quad \psi(u) = \int_{\Omega} V(x) u^2 dx, \quad u \in C_0^\infty(\Omega).$$

Assuming that  $Q_V \geq 0$ , one says that  $\varphi \neq 0$  is a *ground state* of the quadratic form  $Q_V$  if there exists a sequence  $u_k \in C_0^\infty(\Omega)$ , convergent to  $\varphi$  in  $H_{\text{loc}}^1(\Omega)$ , such that  $Q_V(u_k) \rightarrow 0$ . Ground states are sign definite and, up to a constant multiple, unique in the class of positive solutions (that is, positive solutions without global integrability requirements or boundary conditions). If, additionally,  $\varphi \in H_0^1(\Omega)$ , then  $\varphi$  is a minimizer for the Rayleigh quotient

$$\inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\|\nabla u\|_2^2}{\int_{\Omega} V(x) u^2 dx}.$$

There are ground states, however, for which  $\|\nabla \varphi\|_2 = \infty$ . This is the case, in particular, for the ground state  $\varphi(x) = \sqrt{\log \frac{1}{|x|}}$  in the case of Leray potential

$$Q_V = \int_B |\nabla u|^2 dx - \int_B V_{\text{Leray}} u^2 dx.$$

(Leray inequality, [8], states that this form is nonnegative.) Similarly, Hardy inequality in  $\mathbb{R}^N$ ,  $N \geq 3$ , with the radial potential  $V_N$  admits a ground state  $\varphi(x) = |x|^{\frac{2-N}{2}}$ , whose gradient norm is infinite as well.

Existence of a ground state is connected to the property of weak coercivity. The form (2.1) is called weakly coercive if there exists an open set  $E$  relatively compact in  $\Omega$  and a constant  $\delta > 0$ , such that

$$Q_V(u) \geq \delta \left( \int_E u dx \right)^2, \quad u \in C_0^\infty(\Omega).$$

An equivalent criterion of weak coercivity (see [22]) is a seemingly stronger condition that there exists a continuous function  $W > 0$  such that

$$Q_V(u) \geq \int_{\Omega} W(x) (|\nabla u|^2 + u^2) dx, \quad u \in C_0^\infty(\Omega).$$

It is well known that the form (2.1) is nonnegative if and only if it admits a positive solution. However, not any positive solution is a ground state, and in fact, existence of a ground state and weak coercivity for a nonnegative form are mutually exclusive.

**Theorem 2.1.** (*Ground state alternative of Murata [17, 20]*) *A nonnegative functional (2.1) is either weakly coercive or has a ground state.*

If the form (2.1) is nonnegative (and thus admits a positive solution  $v$ ) it can be represented as an integral of a positive function. This representation is known as *ground state transform* or *Jacobi identity*:

$$\int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} V(x) u^2 dx = \int_{\Omega} v^2 \left| \nabla \frac{u}{v} \right|^2 dx.$$

**Remainder in the Trudinger-Moser inequality, radial case.**

**Definition 2.2.** We say that a radial function on the unit disk  $V(|x|) \in \mathcal{V}$  if  $V(r)$  is a nonnegative continuous function on  $(0, 1)$  and the function  $r \mapsto (1 - r^2)^2 V(r)$  is nonincreasing.

**Lemma 2.3.** *If  $V \in \mathcal{V}$  then*

$$(2.2) \quad \sup_{u \in H_0^1(B), Q_V(u) \leq 1} \int_B e^{4\pi u^2} dx = \sup_{u \in H_{0,\text{rad}}^1(B), Q_V(u) \leq 1} 2\pi \int_B e^{4\pi u(r)^2} r dr.$$

*Proof.* Consider  $B$  as the Poincaré disk representing the hyperbolic plane  $\mathbb{H}^2$ . The quadratic form of Laplace-Beltrami operator on  $\mathbb{H}^2$  in the Poincaré disk coordinates is  $\int_B |\nabla u|^2 dx$ . Let  $u^\#$  denote the spherical decreasing rearrangement of  $u \in H_0^1(B)$  relative to the Riemannian measure of the Poincaré disk,  $d\mu = \frac{4dx}{(1-r^2)^2}$ , and recall that the Hardy-Littlewood and the Polia-Szegö inequalities relative to these rearrangements remain valid ([5]). In particular, by the Hardy-Littlewood inequality,

$$\begin{aligned} \int_B V(|x|) u(x)^2 dx &= \int_B \frac{1}{4} (1 - |x|^2)^2 V(|x|) u(x)^2 d\mu \\ &\leq \int_B \frac{1}{4} (1 - r^2)^2 V(r) u^\#(r)^2 d\mu = \int_B V(r) u^\#(r)^2 dx, \end{aligned}$$

and thus, taking into account the Polia-Szegö inequality, we have  $Q_V(u) \geq Q_V(u^\#)$ . From this and the “hyperbolic” Hardy-Littlewood inequality applied to  $\int e^{4\pi u^2} dx$  it follows that the right hand side in (2.2) is not less than the left hand side, while the converse is trivial.  $\square$

**Theorem 2.4.** *Let  $N = 2$ , let  $V \in \mathcal{V}$ , and assume that, for some  $\alpha > 0$ ,*

$$(2.3) \quad \lim_{r \rightarrow 0} r^2 (\log \frac{1}{2})^{2+\alpha} V(r) = 0.$$

Then the quantity

$$S_V = \sup_{u \in H_0^1(B), Q_V(u) \leq 1} J(u), \quad J(u) = \int_B e^{4\pi u^2} dx,$$

is finite if and only if the quadratic form  $Q_V$  is weakly coercive.

*Proof. 1. Necessity.* Assume that  $Q_V$  is not weakly coercive. If  $Q_V(w) < 0$  for some  $w \in H_0^1(B)$ , then  $J(kw) \rightarrow \infty$  and thus  $S_V = +\infty$ . Assume now that  $Q_V \geq 0$ . Then by the ground state alternative,  $Q_V$  has a ground state  $\varphi > 0$  approximated by a  $C_0^\infty$ -sequence  $u_k \rightarrow \varphi$  in  $H_{\text{loc}}^1(B)$  such that  $Q_V(u_k) \rightarrow 0$ . Then, noting that there exist  $\epsilon > 0$  and  $\delta > 0$ , such that for each  $k$ , inequality  $u_k \geq \epsilon$  holds on some set of measure larger than  $\delta$ , we have  $J(u_k/\sqrt{Q_V(u_k)}) \rightarrow \infty$ , which again yields  $S_V = +\infty$ . (Of course,  $Q_V(u_k) \neq 0$  since otherwise  $u_k$  equals  $\varphi$  up to a constant multiple, which is a contradiction since  $\varphi > 0$  and  $u_k \in C_0^\infty(B)$ .)

*2. Sufficiency.* Assume that  $Q_V$  is weakly coercive. By Lemma 2.3 it suffices to consider the problem restricted to radial decreasing functions. Since  $Q_V$  is nonnegative, equation  $Q'_V(u) = 0$  has a positive radial  $C^1$ -solution  $\varphi$ . The latter fact can be inferred from the fact that  $V$ , by (2.3), belongs to the local Kato class  $\mathcal{K}_2$  (see [1]). Let us normalize  $\varphi$  by dividing it by  $\varphi(0)$ , so that  $\varphi(0) = 1$  and  $\varphi(r) \leq 1$ . Define now

$$(2.4) \quad s(r) = e^{\int_1^r \frac{dt}{t\varphi(t)^2}}, \quad 0 < r < 1,$$

so that the function  $s(r)$  satisfies

$$\frac{s'(r)}{s(r)} = \frac{1}{r\varphi(r)^2}.$$

Since  $\varphi(0) = 1$ , we have  $s(r) = \gamma r + o_{r \rightarrow 0}(r)$  with some  $\gamma > 0$ , which implies that  $s(r)$  defines a monotone  $C^1$ -homeomorphism between  $[0, 1]$  and  $[0, s(1))$ , where  $s(1) = \lim_{r \rightarrow 1} s(r)$  may be, generally speaking, infinite. Let  $w : [0, s(1)) \rightarrow [0, 1]$  be the function

$$(2.5) \quad w(s(r)) = u(r)/\varphi(r)$$

Then, writing  $Q_V$  in the ground state transform form and changing the radial integration variable from  $r$  to  $s(r)$  we get

$$Q_V(u) = \int_{B_{s(1)}} |w'(|x|)|^2 dx.$$

Assume first that  $s(1) < \infty$ . Then, taking into account that  $\varphi \leq 1$  and  $r \leq s(r)/s(1)$  (which is easy to infer from (2.4)), we have

$$S_V \leq \sup_{\int_{B_{s(1)}} |\nabla w|^2 = 1} \int_{B_{s(1)}} e^{4\pi\varphi(r(s))^2 w(s)^2} s ds d\theta \leq \sup_{\int_{B_{s(1)}} |\nabla w|^2 = 1} \int_{B_{s(1)}} e^{4\pi w^2} dx < \infty,$$

which proves the theorem in this case. Assume now that  $s(1) = +\infty$ .

Then  $Q_V(u) = \int_{\mathbb{R}^2} |\nabla w|^2 dx$ . Let  $w_k(s) = 1$  for  $r < k$ ,  $w_k(s) = \frac{\log \frac{k^2}{s}}{k}$  for  $k \leq s < k^2$ ,  $w_k(s) = 0$  for  $s \geq k^2$ . Then the sequence  $\varphi(r)w_k(s(r))$  fulfills the definition of approximating sequence for the ground state  $\varphi$  of  $Q_V$ . This, however, in view of the ground state alternative, contradicts the assumption that  $Q_V$  is weakly coercive. Thus  $s(1) < \infty$ , in which case the theorem is already proved.  $\square$

**Example 2.5.** (a) Adimurthi and Druet, [3]: the constant potential  $V(r) = \lambda < \lambda_1$ ; where  $\lambda_1$  is the first eigenvalue of the Dirichlet Laplacian, satisfies the assumptions of Theorem 2.4.

(b) Potential  $V_{\text{Leray}}(r) = \frac{1}{4r^2(\log \frac{1}{r})^2}$  gives  $S_V = +\infty$ , since  $Q_{V_{\text{Leray}}}$  has a ground state  $\varphi(r) = \sqrt{\log \frac{1}{r}}$ .

(c) Another potential satisfying the assumptions of Theorem 2.4 is

$$V_\gamma(r) = \frac{1}{4r^2(\log \frac{1}{r})^2 \max\{(\log \frac{1}{r})^\gamma, 1\}}, \gamma \in (0, \frac{4}{e^2-1}).$$

Since  $V_\gamma < V_{\text{Leray}}$  with the strict inequality on  $(0, e^{-1})$ ,  $Q_V$  is weakly coercive. The potential  $V(r) = \frac{1}{(1-r^2)^2}$ , for which inequality (0.1) was proved in [24], is smaller than  $V_\gamma(r)$ , which (or comparison with the Hardy inequality) implies that  $V_\gamma(r)$  has the optimal multiplicative constant and that the set  $\{Q_{V_\gamma}(u) \leq 1\}$  is not bounded in  $H_0^1(B)$ .

### 3. THE NON-RADIAL CASE AND THE $L^p$ - REMAINDER.

We start with an elementary extension of the result of the previous section to the general bounded domain. We recall that  $w^\#$  denotes rearrangement with respect to the Riemannian measure on the hyperbolic plane.

**Theorem 3.1.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain,  $R = \sqrt{\frac{|\Omega|}{\pi}}$ ,  $V \in L^1_{\text{loc}}(\Omega)$ , and let*

$$\tilde{V}(r) = \frac{[(1 - |x|^2/R^2)^2 V(\frac{x}{R})]^\#(r)}{(1 - r^2)^2}.$$

**Theorem 3.2.** *Assume that  $\tilde{V} \in \mathcal{V}$  and satisfies (2.3), with some  $\alpha > 0$ . If the form  $Q_{\tilde{V}} : H_{0,\text{rad}}^1(B) \rightarrow \mathbb{R}$ , defined as in (2.1), is weakly coercive, then*

$$S_V = \sup_{u \in C_0^\infty(\Omega) : Q_V(u) \leq 1} \int_{\Omega} e^{4\pi u^2} dx < \infty.$$

*Proof.* Rescale the problem to a domain of the area  $\pi$ . Reduce the problem to the radial problem on a unit disk by using rearrangements with respect to the Riemannian measure of  $\mathbb{H}^2$  and apply Theorem 2.4.  $\square$

For the rest of the section we consider the maximization problem

$$S_{\lambda,p} = \sup_{u \in C_0^\infty(\Omega) : Q_{\lambda,p}(u) \leq 1} \int_{\Omega} e^{4\pi u^2} dx < \infty,$$

where

$$Q_{\lambda,p}(u) = \int_{\Omega} |\nabla u|^2 dx - \lambda \|u\|_p^2,$$

and  $\Omega \subset \mathbb{R}^2$ . We will use the following constant:

$$\lambda_p = \inf_{u \in C_0^1(\Omega^*) : \|u\|_p = 1} \int_{\Omega^*} |\nabla u|^2 dx, \quad p > 0,$$

where  $\Omega^*$  is the open ball of radius  $\sqrt{\frac{|\Omega|}{\pi}}$ .

**Theorem 3.3.** *Let  $2 < p < \infty$  and  $\lambda < \lambda_p$ . Then*

$$S_{\lambda,p} = \sup_{u \in C_0^\infty(\Omega) : Q_{\lambda,p}(u) \leq 1} \int_{\Omega} e^{4\pi u^2} dx < \infty.$$

*Proof.* It suffices to verify the assertion in restriction to positive radial decreasing  $H_0^1$ -functions on  $\Omega^*$  when  $\Omega^*$  is the unit disk  $B$ . Let us represent  $Q_{\lambda,p}(u)$  as  $Q_{V_u}(u)$  with  $V_u(u) = \lambda \frac{u^{p-2}}{\|u\|_p^{p-2}}$ ,  $u \in H_{0,\text{rad}}^1$ . Observe that by Hölder inequality

$$\int_B u^{p-2} \varphi^2 dx \leq \|u\|_p^{p-2} \|\varphi\|_p^2,$$

and therefore  $Q_{V_u}(\varphi) \geq Q_{\lambda,p}(\varphi) \geq 0$ . Consequently, there exists a positive radial solution  $\varphi_u$  to the linear equation  $-\Delta \varphi = V_u \varphi$  in  $B$ .



Since, by the standard radial estimate,  $V_u(r) \leq C(\log \frac{1}{r})^{\frac{p-2}{p}}$ , one has  $\varphi_u \in C^1(B)$ , and the maximum of  $\varphi_u$  is at the origin. We assume without loss of generality that  $\varphi_u(0) = 1$ . By the ground state transform we have for any  $v \in C_0^\infty(B)$ ,

$$Q_{V_u}(v) = \int_B \varphi_u^2 \left| \nabla \frac{v}{\varphi_u} \right|^2 dx, \quad v \in C_0^\infty(B).$$

Let now

$$s_u(r) = e^{\int_{e^{-1}}^r \frac{dt}{t\varphi_u(t)^2}}, \quad 0 < r < 1,$$

and note that this function satisfies

$$\frac{s'_u(r)}{s_u(r)} = \frac{1}{r\varphi_u(r)^2}.$$

Observe that since  $\varphi_u(0) = 1$  and  $\varphi_u$  is a classical solution, we have  $s_u(r) = \gamma r + o_{r \rightarrow 0}(r)$  with some  $\gamma > 0$ , and thus the mapping  $r \mapsto s_u(r)$  is a monotone  $C^1$ -homeomorphism between  $[0, 1)$  and  $[0, s_u(1))$ . We will show now that  $\varphi_u$  is bounded away from zero near  $r = 1$ , uniformly in a  $H_{0,\text{rad}}^1(B)$ -ball of  $u$ . First note that if for some  $u \in H_{0,\text{rad}}^1(B)$  one has  $\varphi_u(1) = 0$ , then  $\varphi_u$  is the first eigenfunction for the Dirichlet eigenvalue problem  $-\Delta\varphi = V_u\varphi$  in  $B$ . From the Hölder inequality and the definition of  $\lambda_p$  we get:

$$\begin{aligned} \int_B |\nabla \varphi_u|^2 dx &= \int_B V_u \varphi^2 dx \leq \lambda \left( \int_B \left( \frac{u}{\|u\|_p} \right)^p \right)^{1-2/p} \left( \int_B \varphi_u^p \right)^{2/p} \\ &\leq \lambda \lambda_p^{-1} \int_B |\nabla \varphi_u|^2 dx < \int_B |\nabla \varphi_u|^2 dx, \end{aligned}$$

a contradiction. Thus  $\varphi_u(1) > 0$  for any  $u$ , and it remains to show that  $\varphi_u(r)$  has a common positive lower bound for all  $u$  and all  $r$  near 1. Indeed, assume that there is a sequence  $u_k$  with  $Q_{\lambda,p}(u_k) \leq 1$ , and a sequence  $r_k \rightarrow 1$  such that  $\varphi_{u_k}(r_k) \rightarrow 0$  and  $-\Delta\varphi_{u_k} = \lambda u_k^{p-2} \varphi_{u_k}$ . Note that since  $\lambda < \lambda_p$ , the sequence  $u_k$  is bounded in  $H_0^1(B)$ , and without loss of generality we may assume that  $u_k \rightharpoonup u$  in  $H_0^1(B)$  with  $Q_{\lambda,p}(u) \leq 1$ . From here one can easily derive that  $\varphi_{u_k}$  converges uniformly to some nonnegative  $\varphi$  with  $\varphi(1) = 0$ , and that  $\varphi$  satisfies the equation  $-\Delta\varphi = V_u\varphi$ . In other words,  $\varphi = \varphi_u$  and we have  $\varphi_u(1) = 0$ , which is a contradiction. We conclude that there exists  $\epsilon > 0$  and  $\delta > 0$ , such that  $\inf_{r \in [1-\epsilon, 1], u: Q_{\lambda,p}(u) \leq 1} \varphi_u(r) \geq \delta$ . This implies that there is a number  $S$  such that  $s_u(1) \leq S$  for all  $u$  satisfying  $Q_{\lambda,p}(u) \leq 1$ .

For each  $v \in H_{0,\text{rad}}^1(B)$  define the following function on  $[0, s_u(1))$ :

$$w_{v;u}(s_u(r)) = v(r).$$

Then, applying the ground state transform and the changing the radial integration variable from  $r$  to  $s_u$ , we have

$$Q_{V_u}(v) = \int_B \varphi_u^2 \left| \nabla \frac{v}{\varphi_u} \right|^2 dx = \int_{B_{s_u(1)}} |w'_{v;u}(|x|)|^2 dx, \quad v \in H_{0,\text{rad}}^1(B).$$

By setting  $v = u$ , we get from here

$$Q_{\lambda,p}(u) = \int_{B_{s_u(1)}} |w'_{u;u}(|x|)|^2 dx, \quad v \in H_{0,\text{rad}}^1(B).$$

Then, taking into account that  $\varphi_u \leq 1$  for every  $u$ , we arrive at

$$S_{\lambda,p} \leq S^2 \sup_{\int_B |\nabla w|^2 = 1} \int_B e^{4\pi w(|x|)^2} dx < \infty.$$

which proves the theorem.  $\square$

#### 4. RELATED INEQUALITIES

The arguments in Sections 2 and 3 allow to give the following refinement of the Onofri-Beckner inequality ([17, 6]). The original inequality for the unit disk is

$$(4.1) \quad \log \left( \frac{1}{\pi} \int_B e^u dx \right) + \left( \frac{1}{\pi} \int_B e^u dx \right)^{-1} \leq 1 + \frac{1}{16\pi} \int_B |\nabla u|^2 dx, \quad u \in C_0^\infty(B).$$

**Theorem 4.1.** *Let  $\Omega = B$  and assume that  $\psi(u) = \int_B V u^2 dx$  with  $V$  as in Theorems 2.4 and 3.1, or that  $\psi(u) = \lambda \|u\|_p^2$ ,  $\lambda < \lambda_p$ ,  $p > 2$ , as in Theorem 3.3. Then for every  $u \in C_0^\infty(B)$ ,*

$$(4.2) \quad \log \left( \frac{1}{\pi} \int_B e^u dx \right) + \left( \frac{1}{\pi} \int_B e^u dx \right)^{-1} \leq 1 + \frac{1}{16\pi} \left( \int_B |\nabla u|^2 dx - \psi(u) \right).$$

*Proof.* We give the proof for the case of the remainder term  $\psi$  as in Theorem 2.4. The proofs in other cases are analogous. By the standard rearrangement argument it suffices to consider the radially symmetric functions.

Assume first that  $u \geq 0$ . Without loss of generality we may assume that  $u$  is radial. Let us use the coordinate transformation (2.4) and

the substitution (2.5). Taking into account that the function  $F(t) := \log t + t^{-1}$  is increasing on  $(1, \infty)$ , that the function  $\varphi$ , involved in the transformation, does not exceed 1, and that, as it is immediate from (2.4),  $s(r)/s(1) \geq r$  we have from (4.1)

$$\begin{aligned} F\left(\frac{1}{\pi s(1)^2} \int_{B_{s(1)}} e^{\varphi(r(s))w(s)} \frac{r(s)^2 \varphi(r(s))^2}{s^2} dx(s)\right) &\leq \\ &\leq 1 + \frac{1}{16\pi} \int_{B_{s(1)}} |\nabla w|^2 dx, \quad w \in H_{0,rad}^1(B_{s(1)}). \end{aligned}$$

Using (2.5) in order to return to the original variable  $u$ , we immediately have (4.2) for  $u \geq 0$ .

Consider now the case  $u \leq 0$ . Without loss of generality we again assume that  $u$  is radial. Then, taking into account (2.4), (2.5),  $\varphi \leq 1$ ,  $r \leq s(r)$ , and the fact that the function  $F$  is decreasing on  $(0, 1)$ , we have

$$\begin{aligned} F\left(\frac{1}{\pi} \int_B e^u dx\right) &\leq F\left(\frac{1}{\pi} \int_B e^{w(s(r))} dx\right) \\ &= F\left(\frac{1}{\pi s(1)^2} \int_{B_{s(1)}} e^{w(s)} \frac{s^2}{r(s)^2 \varphi(r(s))^2} dx(s)\right) \\ &\leq F\left(\frac{1}{\pi s(1)^2} \int_{B_{s(1)}} e^{w(s)} dx(s)\right) \\ &\leq 1 + \frac{1}{16\pi} \int_{B_{s(1)}} |\nabla w|^2 dx = 1 + \frac{1}{16\pi} Q_V(u). \end{aligned}$$

Finally, we write a general  $u$  as  $u = u^+ + (-u^-)$  and note that the function  $\log t + 1/t$  is subadditive on  $(0, \infty)$ . We leave it to the reader to prove the subadditivity with help of the following sketch: collect the logarithmic terms in the subadditivity inequality into a single logarithm, invert the logarithm, and replace the resulting exponential function by its Taylor polynomial up to the order 2. Inequality (4.2) is then immediate from the cases where  $u \geq 0$  and  $u \leq 0$ .  $\square$

**Corollary 4.2. (*Inequality of Adimurthi-Druet type.*)** *Let  $Q(u) = \|\nabla u\|_2^2 - \psi(u)$  be any of the functionals  $Q_V$  as in Theorems 2.4 and 3.1, or the functional  $Q_P$ , as in Theorem 3.3. Then*

$$\sup_{\|\nabla u\|_2 \leq 1} \int_{\Omega} e^{4\pi(1+\psi(u))u^2} dx \leq \sup_{\|\nabla u\|_2 \leq 1} \int_{\Omega} e^{\frac{4\pi u^2}{1-\psi(u)}} dx < \infty$$

*Proof.* Note first that the integral in the left hand side is smaller than the integral in the right hand side by the inequality  $(1 + \psi)(1 - \psi) < 1$ . Let  $u = \sqrt{\gamma}v$  with  $\|\nabla v\|_2 = 1$ . Then  $Q(u) \leq 1$  is equivalent to  $\gamma - \gamma\psi(v) \leq 1$ , i.e.  $\gamma \leq \frac{1}{1-\psi(v)}$ . Write (0.1), substitute  $u^2 = \gamma v^2$  into the integral and rename  $v$  as  $u$ .  $\square$

**Corollary 4.3.** *Let  $\|\cdot\|_{\text{Orl}}$  denote the Orlicz norm associated with the Trudinger-Moser functional on a bounded domain  $\Omega \subset \mathbb{R}^2$ , and let  $Q(u) = \|\nabla u\|_2^2 - \psi(u)$  be any of the functionals  $Q_V$  as in Theorems 2.4 and 3.1, or the functional  $Q_p$ , as in Theorem 3.3. Then there exists a  $C > 0$  such that*

$$\int_{\Omega} |\nabla u|^2 dx - \psi(u) \geq C \|u\|_{\text{Orl}}^2$$

*Proof.* Assume first that  $Q(u) = 1$ . From the uniform bound on  $\int_{\Omega} (e^{4\pi u^2} - 1) dx$  in (0.1) follows a uniform bound for the Orlicz norm, which yields the inequality under the constraint  $Q(u) = 1$ . It remains to use the standard homogeneity argument.

### Open problems.

- (1) Does the inequality (0.1) hold for general bounded  $\Omega$ , all potentials  $V$  of the local Kato class  $\mathcal{K}_2$  and all  $p \in (0, \infty)$ , as long as the constraint functional  $Q$  remains weakly coercive?
- (2) When  $\Omega = \mathbb{R}^2$ , inequality (0.1) with  $Q(u) = \|\nabla u\|_2^2$  is false, since the form  $\|\nabla u\|_2^2$  on the whole  $\mathbb{R}^2$  admits a ground state 1. On the other hand, the inequality holds when  $Q(u) = \|\nabla u\|_2^2 + \|u\|_2^2$  (Ruf, [21]). Furthermore, as it is shown in [10], inequality (0.1) with  $Q(u) = \|\nabla u\|_2^2$  holds for a simply connected (generally unbounded) domain  $\Omega \subset \mathbb{R}^2$  if and only if  $\|\nabla u\|_2^2 \geq \lambda \|u\|_2^2$  with some  $\lambda > 0$ . In both results the condition is  $L^2$ -coercivity,  $Q(u) \geq C \|u\|_2^2$ . It is natural then to ask, for unbounded domains, if there are weaker coercivity conditions on  $Q$  that yield (0.1)?
- (3) Since Hardy-Sobolev-Maz'ya inequalities can be derived from Caffarelli-Kohn-Nirenberg inequalities ([7]) via the ground state transform, it is natural to ask what could be an analog of Caffarelli-Kohn-Nirenberg inequalities related to the remainder estimates of the Hardy-Moser-Trudinger type.
- (4) Our reduction to the radial case is of tentative character, as it is based on rearrangements specific to the hyperbolic plane which resulted in a restrictive condition of weighted monotonicity on the potential. Perhaps more general rearrangements satisfying

Polia-Szegö and Hardy-Littlewood inequalities (see [13]) can be used to relax the monotonicity condition on the potential.

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